On the Rayleigh–Taylor problem in magneto-hydrodynamics with finite resistivity

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In order to elucidate the importance of the infinite conductivity assumption in MHD a simple problem has been studied. This is a Rayleigh–Taylor problem of two superposed fluids under gravity partially stabilized by a uniform, horizontal magnetic field. It is found that the inclusion of a small, but finite resistivity introduces new and unexpected solutions. For instance, moderately long, 'stabilized' waves are now found to grow aperiodically and unexpectedly rapidly at a rate $\propto$ (resistivity)$^{\frac{1}{3}}$. Other modes are found to be periodic and damped at a rate $\propto$ (resistivity)$^{\frac{1}{2}}$.

1. Introduction

It has been conjectured that the assumption of infinite electrical conductivity for a fluid, apart from the neglect of other dissipative processes, may yield serious discrepancies between theoretical predictions of idealized magnetohydrodynamics (MHD) and experiment (Tayler 1960; Jukes 1961). For example, there are the stability properties of laboratory plasmas in the hard core geometry (Bickerton *et al.* 1961). Mathematically speaking the omission of even the smallest resistivity reduces the order of the equations and experience should make us wary lest in such circumstances mathematical solutions give a misleading understanding of physical reality. We therefore consider it important to solve a simple MHD stability problem which does include finite resistivity and to compare these new solutions with the old ones of idealized MHD theory. It may not matter for this restricted purpose that the model is very simple, even to the extent of having no practical or experimental consequence. Of course, an ultimate goal would be to find a model of a situation sufficiently simple to analyse, which could also be subject to experimental test.

The problem discussed at length in this paper is a modification of the so-called Rayleigh–Taylor problem (Chandrasekhar 1961), which in essence is the stability of two fluids of different densities supported one on the other and subjected to an acceleration, or in a vertical gravitational field. We shall consider the case when one of the fluids is electrically conducting and the system is placed in a uniform, horizontal magnetic field. If the fluid which is uppermost is also the heavier the system is potentially gravitationally unstable, but if it is also electrically conducting it may also be stable to waves of sufficiently short length propagating *along* the field lines. Physically this is because the tubes of magnetic flux are constrained to move with the fluid and to bend them requires energy. In fact, idealized MHD theory predicts complete stability for all wavelengths shorter than a critical calculable value. Waves which propagate *across* the field lines and which we do not consider here, may not be required to bend the flux tubes and so may be unstable. For this reason, amongst others, it may not be possible to perform an actual experiment on so simple a system.

We further assume for simplicity that the fluids are incompressible, inviscid and with no surface tension at their interface. One does not expect the first and third assumptions to omit any vital physics. The second assumption is more seriously unrealistic, since viscosity would obviously influence the growth or decay rates of perturbations. However, one would not expect viscous dissipation to eliminate an aperiodically growing perturbation. In any case, we are not attempting here a full understanding of a physically realizable situation but merely examining shortcomings in an existing assumption.

One finds that the introduction of higher order terms radically changes the possible solutions. With small, but finite resistivity, new branches are found corresponding in one instance to aperiodically growing disturbances at wavelengths which appeared to be stabilized according to the predictions of idealized MHD theory.

2. The model and the equations of the problem

We consider a partially conducting fluid of density ρ_1 and resistivity η occupying the half space $0 > z > -\infty$ and supported by a non-conducting fluid of density ρ_2 occupying the half space $+\infty > z > 0$. Both are placed in a horizontal magnetic field $\mathbf{B} = (B, 0, 0)$ and a vertical gravitational field $\mathbf{g} = (0, 0, g)$ as in figure 1. The equilibrium fluid pressure p, which satisfies the hydrostatic equation $\rho\mathbf{g} = \operatorname{grad} p$, is assumed to be always positive and to support the fluid. Thus for $z < 0$, $p = \rho_1 gz + p_c$ where p_c can be made an arbitrarily large constant to satisfy $p > 0$ as $z \to -\infty$ as far as we please. The absolute magnitude of p is irrelevant in this problem.

Let us consider perturbations of the equilibrium of the form

$$\operatorname{Re} f(z) \exp i(kx + \omega t).$$

Note that we are deliberately restricting our discussion to modes with wave vector $\mathbf{k}$ satisfying $\mathbf{k} \times \mathbf{B} = 0$. For these modes certain components of the perturbed quantities vanish and we shall denote the remaining components as follows: velocity $(u, 0, w)$, magnetic field $(B_x, 0, B_z)$, fluid pressure δp (scalar). The equations of motion and continuity for the conducting fluid (suffix 1) are

$$\rho_1 i\omega w_1 = -D\,\delta p_1 - B(DB_{x_1} - ikB_{z_1}), \tag{1}$$

$$\rho_1 i\omega u_1 = -ik\,\delta p_1, \tag{2}$$

and
$$Dw_1 + iku_1 = 0, \tag{3}$$

where $D \equiv \partial/\partial z$. In the conducting fluid we shall assume a generalized form of Ohm's law given by

$$\eta \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \nabla\phi, \tag{4}$$

where $\mathbf{j}$ is the current, $\mathbf{E}$ the electric field. ϕ may be any scalar function of, say, pressure. We assume that the resistivity is constant and isotropic and we neglect displacement currents. By taking the curl of equation (4) and using the Maxwell equations (in rationalized e.m. units)

$$\operatorname{curl}\mathbf{B} = \mathbf{j}, \tag{5}$$

$$\operatorname{curl}\mathbf{E} = -\partial/\partial t\,\mathbf{B}, \tag{6}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{7}$$

we can obtain

$$i\omega B_{z_1} - ikw_1 B = \eta(D^2-k^2)B_{z_1}, \tag{8}$$

$$i\omega B_{x_1} + Dw_1 B = \eta(D^2-k^2)B_{x_1}, \tag{9}$$

$$DB_{z_1} + ikB_{x_1} = 0. \tag{10}$$

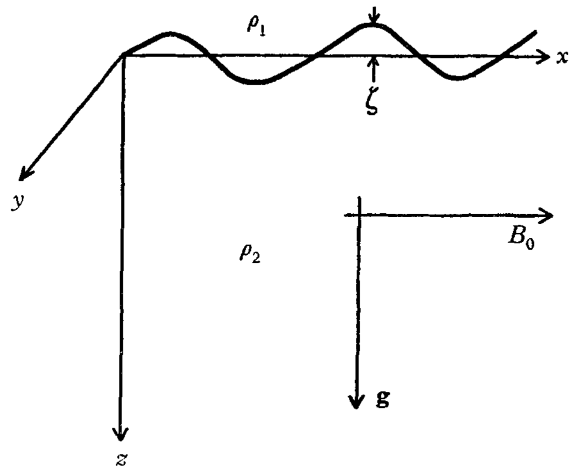

FIGURE 1. The geometry of the problem.

Thus, from (1), (2), (3), (10),

$$(D^2-k^2)w_1 = (kB/\omega\rho_1)(D^2-k^2)B_{z_1}. \tag{11}$$

Rearranging (8),

$$ikw_1 = B^{-1}[i\omega - \eta(D^2-k^2)]B_{z_1}. \tag{12}$$

In the non-conducting fluid (suffix 2) there is no current, so that

$$\mathbf{j}_2 = \operatorname{curl}\mathbf{B}_2 = DB_{x_2} - ikB_{z_2} = 0. \tag{13}$$

Also

$$\operatorname{div}\mathbf{B}_2 = 0 = DB_{z_2} + ikB_{x_2}. \tag{14}$$

These equations and the equations of motion for the non-conductor ((1), (2), (3) with $\mathbf{j} \equiv 0$) give

$$(D^2-k^2)B_{z_2} = 0, \tag{15}$$

$$(D^2-k^2)w_2 = 0. \tag{16}$$

In both fluids from (2) and (3)

$$\delta p_j = -ik\rho_j(\omega/k^2)Dw_j \quad (j=1,2). \tag{17}$$

Along the *perturbed* fluid interface, pressures in both fluids are equal as well as their vertical displacement ζ. Because the densities are discontinuous at the interface, gravity forces occur in the displacement. Hence

$$(\delta p_1)_0 - B\int_0^\zeta j_{y_1}\,dz + \rho_1\zeta g = (\delta p_2)_0 + \rho_2\zeta g, \tag{18}$$

where

$$i\omega\zeta = w_{12} = w_1 = w_2, \tag{19}$$

and $(\delta p_j)_0$ is the pressure perturbation of the fluids evaluated at $z = 0$. The second term of (18), the electromagnetic force across the displacement, is of second-order smallness and negligible in the case of finite resistivity. It has been included for illustration because it occurs in infinite-conductivity theory where there is a finite skin current of vanishing thickness, the product of current *density* and thickness remaining finite. At $z = 0$ combining (17), (18) and (19)

$$(\rho_1 Dw_1 - \rho_2 Dw_2)\, i\omega/k^2 = (\rho_1 - \rho_2)\, g w_{12}/i\omega. \tag{20}$$

Other boundary conditions across the interface in addition to (19) and (20) are that (i) B_z is continuous, (ii) DB_z is continuous (because B_x is). Also (iii) perturbations must be bounded as $|z| \to \infty$.

3. The deduction of the dispersion equation

Because of obvious symmetries there can be no loss of generality in assuming that $k \geqslant 0$ and $\mathrm{Re}\,\omega \geqslant 0$. Equations (11) and (12) can be completely solved in the form

$$B_{z_1}/B = A_1 e^{\alpha z} + A_2 e^{kz}, \tag{21}$$

$$w_1/V = A_3 e^{\alpha z} + A_4 e^{kz}, \tag{22}$$

where

$$V^2 \equiv B^2/\rho_1. \tag{23}$$

The A's are arbitrary constants (to be eliminated) and α is an eigen-value which must be chosen with a positive real part so that physical quantities are bounded as $z \to -\infty$. This boundary condition eliminates the second exponential solutions of opposite sign.

Likewise in the non-conducting fluid we consider only solutions which are obviously bounded as $z \to +\infty$,

$$w_2/V = A_2 e^{-kz}, \tag{24}$$

$$B_{z_2}/B = A_6 e^{-kz}. \tag{25}$$

Substituting (21), (22) back into (12) gives

$$ik(A_3 e^{\alpha z} + A_4 e^{kz})\, V = [i\omega(A_1 e^{\alpha z} + A_2 e^{kz}) - \eta(\alpha^2 - k^2)\, A_1 e^{\alpha z}].$$

Since this equation must hold for all z, both

$$ikVA_3 = [i\omega - \eta(\alpha^2 - k^2)]\, A_1$$

and

$$ikVA_4 = i\omega A_2.$$

Substituting (21) and (22) back into (11) gives

$$\omega A_3 = kVA_1,$$

and (22) and (24) into (19) gives

$$A_3 + A_4 = A_5.$$

The boundary conditions (i), (ii) give

$$A_1 + A_2 = A_6 \quad \text{and} \quad \alpha A_1 + kA_2 = -kA_6.$$

These last six relations are sufficient to eliminate the six A's and to relate α to k, ω as later in (29). The dispersion equation $\omega = \omega(k)$ can be obtained from the interface condition (20). Evaluating first Dw_1/w_{12} and Dw_2/w_{12} at $z = 0$ from (22), (24), (19), as

$$Dw_1/w_{12} = k[\alpha - \tfrac{1}{2}\omega^2(\alpha+k)/k^2V^2]/[k - \tfrac{1}{2}\omega^2(\alpha+k)/k^2V^2], \tag{26}$$

and

$$Dw_2/w_{12} = -k, \tag{27}$$

substituting into (20) gives the required dispersion equation,

$$-(\rho_1-\rho_2)(gk/\omega^2)\{k-\tfrac{1}{2}\omega^2(\alpha+k)/k^2V^2\} = \rho_1\{\alpha-\tfrac{1}{2}\omega^2(\alpha+k)/k^2V^2\}+\rho_2\{k-\tfrac{1}{2}\omega^2(\alpha+k)/k^2V^2\}, \tag{28}$$

where α is the root of

$$\alpha^2 = k^2 + (k^2V^2-\omega^2)/i\omega\eta, \tag{29}$$

which has $\text{Re}(\alpha) > 0$. Let us put the equations (28), (29) into dimensionless form with the substitutions

$$\left.\begin{aligned} &\Omega \equiv \omega V/g, \quad G \equiv g/kV^2, \quad R \equiv V^3/g\eta, \quad V^2 \equiv B^2/\rho_1 \\ \text{and}\quad &\xi \equiv \alpha/k, \quad \text{with} \quad \text{Re}\,\xi > 0 \text{ only.} \end{aligned}\right\} \tag{30}$$

Then

$$\xi^2 = 1 + (1-\Omega^2G^2)R/i\Omega, \tag{31}$$

and the dispersion equation may be conveniently written as

$$\xi = H/K, \tag{32}$$

where

$$H \equiv \rho_2 - \tfrac{1}{2}(\rho_1+\rho_2)\Omega^2G^2 + \tfrac{1}{2}(\rho_1-\rho_2)(2-\Omega^2G^2)/\Omega^2G, \tag{33}$$

$$K \equiv -\rho_1 + \tfrac{1}{2}(\rho_1+\rho_2)\Omega^2G^2 + \tfrac{1}{2}(\rho_1-\rho_2)G. \tag{34}$$

To be definite, we take $\rho_1 > \rho_2$. The potentially unstable case then corresponds to g (or G) > 0 and the gravitationally stable case to g (or G) < 0. In the latter case we shall reverse the sign of V too, so that R remains positive and the sign of Ω is unchanged. A complete solution of the problem is to find Ω (possibly complex, but with $\text{Re}\,\Omega \geqslant 0$) for all real $G\,(\infty > G > -\infty)$ for positive, real k and positive values of the parameter R. We shall consider the potentially unstable case $G > 0$ and then the potentially stable case $G < 0$. In each case we shall give the infinite-conductivity solution and then find the solution for small, but finite resistivity $R \gg 1$ by an expansion treatment of (31) and (32).

(*a*) *The gravitationally unstable case, $G > 0$*

The infinite conductivity solution is obtained by setting $R = \infty$ and therefore $\xi = \infty$, $K = 0$, to give

$$\Omega_0^2 = 2\{\rho_1 - \tfrac{1}{2}G(\rho_1-\rho_2)\}/(\rho_1+\rho_2)G^2, \tag{35}$$

thence

$$G^2\Omega_0^2 - 1 = \left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)(1-G). \tag{36}$$

For $R = \infty$:

Ω_0^2 is real and positive if $G < 2\rho_1/(\rho_1-\rho_2)$, implying a stable oscillation;

Ω_0^2 is real and negative if $G > 2\rho_1/(\rho_1-\rho_2)$, implying a growing and the accompanying decaying, aperiodic mode.

We now consider the most important case when R is large, but remains finite. We shall first show that there exists an aperiodically growing solution even for

$G < 2\rho_1/(\rho_1-\rho_2)$. Let us set $\Omega = -i\beta$ where this solution corresponds to a real, positive β. From (31)

$$\xi = [1+(1+\beta^2G^2)\,R/\beta]^{\frac{1}{2}}. \tag{37}$$

For small $\beta \to 0$,

$$\xi \to R^{\frac{1}{2}}\beta^{-\frac{1}{2}},$$

$$H \to -(\rho_1-\rho_2)/G\beta^2,$$

$$K \to -\tfrac{1}{2}(\rho_1-\rho_2)\,[\{2\rho_1/(\rho_1-\rho_2)\}-G].$$

From the dispersion equation (32) we see that there can be a solution when $G < 2\rho_1/(\rho_1-\rho_2)$, for, while $\xi \to +\infty$ as $R^{\frac{1}{2}}\beta^{-\frac{1}{2}}$, at the same time $H/K \to +\infty$ as β^{-2}. Thus there exists a solution for which $\beta \sim O(R^{-\frac{1}{3}})$.

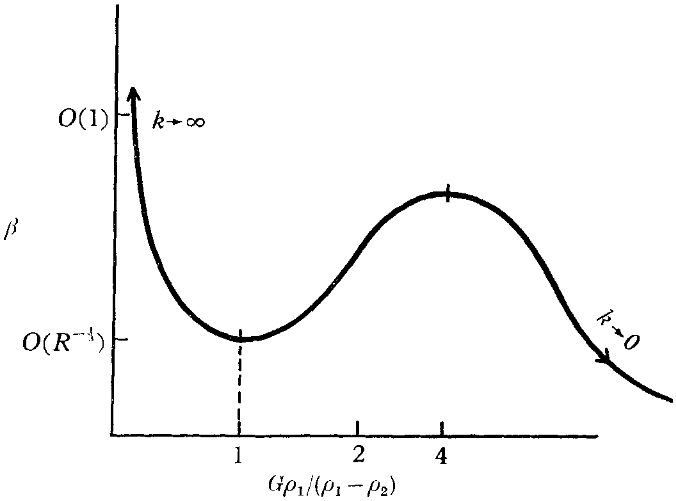

FIGURE 2. Gravitationally unstable case $G > 0$. Sketch of the aperiodic growth rate $\beta = -i\omega V/g$ as a function of wavelength $G = g/kV^2$ for large $R\,(= V^3/g\eta)$. According to infinite conductivity, values along the abscissa < 2 would be stable.

More precisely we can solve (32) by keeping terms of order $R^{\frac{1}{2}}$ in the expression for ξ, (37), and assuming $\beta^2G^2 \ll 1$ and $\beta^2G \ll 1$. It is convenient to solve for G in terms of β to find

$$G = \frac{\rho_1}{\rho_1-\rho_2} \pm \left[\left(\frac{\rho_1}{\rho_1-\rho_2}\right)^2 - \frac{2}{\beta^{\frac{3}{2}}R^{\frac{1}{2}}}\right]^{\frac{1}{2}}. \tag{38}$$

Thus β passes through a minimum $2^{\frac{2}{3}}R^{-\frac{1}{3}}[(\rho_1-\rho_2)/\rho_1]^{\frac{4}{3}}$ for $G = \rho_1/(\rho_1-\rho_2)$. For larger G, β increases, merging with the infinite R solution when

$$G \gtrsim 2\rho_1/(\rho_1-\rho_2),$$

for which $\beta \sim O(1)$. For $G < \rho_1/(\rho_1-\rho_2)$ and tending to zero, β increases as $G^{-\frac{2}{3}}R^{-\frac{1}{3}}$. However, the validity of the present solution fails at $\beta^2G \sim O(1)$, at which point $G \sim O(R^{-2})$ and $\beta \sim O(R)$, while for $G < O(R^{-2})$, $\xi \cong 1$ and $\beta \to G^{-\frac{1}{2}}\{(\rho_1-\rho_2)/(\rho_1+\rho_2)\}^{\frac{1}{2}}$. We can now plot schematically $\beta\ (= +i\Omega)$ as a function of $G(\propto k^{-1})$ for $R \gg 1$ as is done in figure 2. We shall not concern ourselves here with very small values of $G \ll O(R^{-2})$, but see Discussion.

We next examine solutions along the positive imaginary axis of Ω by putting $\Omega = +i\beta$. The solution for large G ($> 2\rho_1/(\rho_1-\rho_2)$) is nearly as before. However, for large, but finite R, ξ is complex. This implies that Ω is complex and that this part of the solution must run close to, but not actually along the imaginary axis. We can obtain a solution for $R \gg 1$ for this branch by a perturbation about the imaginary axis. By a similar perturbation we can show that for smaller values of G this branch continues on close to the real positive axis of Ω (see figure 3).

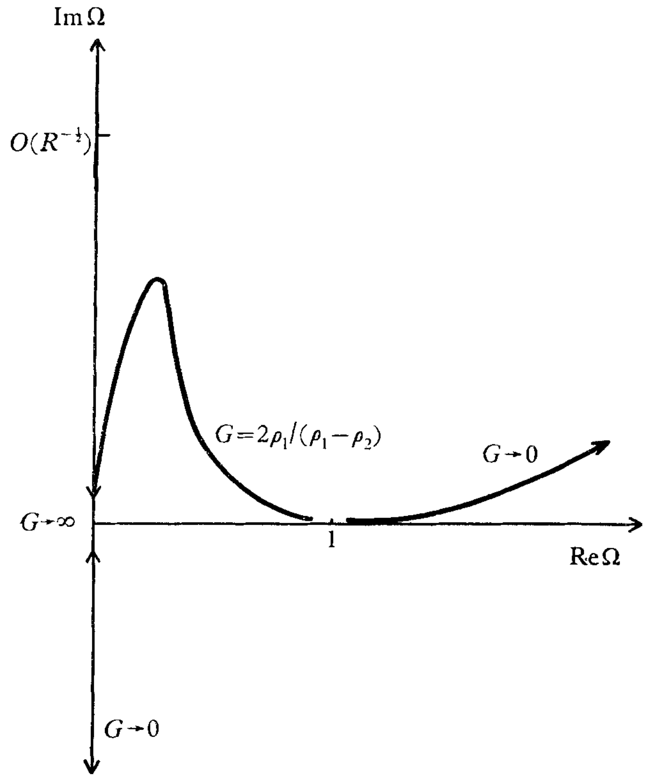

FIGURE 3. Gravitationally unstable case $G > 0$. Sketch of complex growth rate $\Omega = \omega V/g$ as a function of wavelength $G = g/kV^2$ for large R ($= V^3/\eta g$). The aperiodic, growing mode lies along the negative, imaginary axis, and is shown expanded in figure 2.

The large R expansion goes as follows. If we set

$$\Omega = \Omega_0 + \Omega_1 + \ldots, \quad \xi = \xi_0 + \xi_1 + \ldots, \tag{39}$$

where $K(\Omega_0) \equiv 0$, then

$$\Omega_1 = H(\Omega_0)/\xi_0 K'(\Omega_0) \sim O(R^{-\frac{1}{2}}), \tag{40}$$

where a prime denotes differentiation with respect to Ω. We now expand about the positive imaginary axis of Ω and putting $\Omega_0 = i\beta_0$ ($\beta_0 > 0$), where

$$\beta_0^2 G^2 = \left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)\left[G - \frac{2\rho_1}{\rho_1-\rho_2}\right], \tag{41}$$

we note that $G > 2\rho_1/(\rho_1-\rho_2) > 1$. We find

$$H(\Omega_0) = -2\rho_1(G-1)/\{G-2\rho_1/(\rho_1-\rho_2)\}, \quad K'(\Omega_0) = i\beta_0 G^2(\rho_1+\rho_2),$$

and $$\xi_0 = \pm iR^{\frac{1}{2}}(G-1)^{\frac{1}{2}}\{(\rho_1-\rho_2)/(\rho_1+\rho_2)\}^{\frac{1}{2}}\beta_0^{-\frac{1}{2}}.$$

Hence $$\Omega_1 = \frac{2\rho_1}{\rho_1+\rho_2}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)^{\frac{1}{2}}\frac{(G-1)^{\frac{1}{2}}}{G^4}R^{-\frac{1}{2}}\beta_0^{-\frac{5}{2}}, \tag{42}$$

since only the positive real root is relevant. As $G \to \infty$, $\beta_0 \sim O(G^{-\frac{1}{2}}) \to 0$ and $\Omega_1 \to 0$. As $G \to 2\rho_1/(\rho_1-\rho_2)$, $\Omega_1 \to \infty$ and the expansion diverges, indicating that higher order terms are of progressive importance as $\beta_0 \to 0$. We now expand

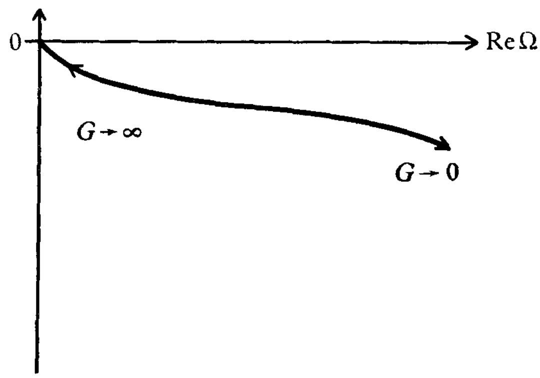

FIGURE 4. Gravitationally stable case $G < 0$. Sketch of Ω in the complex plane as a function of G for large R.

about the positive real axis of Ω (but exluding the neighbourhood of the singular point $\Omega_0 = 1 = G$ where solutions are trivial) and set $K(\Omega_0) \equiv 0$ to give

$$\Omega_0^2 G^2 = \left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)\left[\frac{2\rho_1}{\rho_1-\rho_2} - G\right], \tag{43}$$

where $$H(\Omega_0) = 2\rho_1\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)\left(\frac{G-1}{G^2}\right)\Omega_0^{-2}, \quad K'(\Omega_0) = (\rho_1+\rho_2)\Omega_0 G^2,$$

and $$\xi_0 = \frac{1\pm i}{\sqrt{2}}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)^{\frac{1}{2}}\frac{(|G-1|)^{\frac{1}{2}}}{\Omega_0^{\frac{1}{2}}}, \tag{44}$$

according as $G \gtrless 1$. From equation (40) we see that the $\mathrm{Im}(\Omega_1) \geqslant 0$ always. Only damped oscillations occur. On substitution,

$$\mathrm{Im}(\Omega_1) = \sqrt{2}\frac{\rho_1}{\rho_1+\rho_2}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)^{\frac{1}{2}}\frac{(|G-1|)^{\frac{1}{2}}}{G^4}\frac{R^{-\frac{1}{2}}}{\Omega_0^{\frac{5}{2}}}. \tag{45}$$

As $\Omega_0 \to \infty$, $\mathrm{Im}(\Omega_1) \sim O(G^{-\frac{3}{4}}) \to \infty$. As $\Omega_0 \to 0$, $G \to 2\rho_1/(\rho_1-\rho_2)$ and $\mathrm{Im}(\Omega_1) \to \infty$. The expansion diverges at $\Omega_0 = 0$ and ∞, but at $\Omega_0 = 0$ it merges with the expansion about the positive imaginary axis.

(*b*) *The gravitationally stable case,* $G < 0$

Setting $K(\Omega_0) = 0$ immediately shows that Ω_0 is always real. Only positive values are relevant. It is noteworthy that the singular point no longer occurs. Straightforward manipulation shows that

$$\mathrm{Im}(\Omega_1) = \frac{\sqrt{2}\,\rho_1}{\rho_1+\rho_2}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)^{\frac{1}{2}} \frac{(1+|G|)^{\frac{1}{2}}\,R^{-\frac{1}{2}}}{G^4\Omega_0^{\frac{5}{2}}}. \tag{46}$$

As $|G| \to \infty$, $\mathrm{Im}(\Omega_1) \to 0$. As $|G| \to 0$, $\mathrm{Im}(\Omega_1) \to G^{-\frac{3}{2}}R^{-\frac{1}{2}}$, and diverges. Solutions are shown schematically in figure 4.

4. Discussion

The existence of aperiodic, growing solutions in case (*a*) is not unexpected for the long waves satisfying $G > 2\rho_1/(\rho_1-\rho_2)$ which propagate along the magnetic field, since this is predicted by infinite-conductivity theory. What is unexpected is the emergence of a new branch of the solution for shorter waves

$$G \sim O(\rho_1/(\rho_1-\rho_2))$$

with aperiodic growth rate $|\Omega| \sim O(R^{-\frac{1}{3}})$, or in physical terms

$$|\omega| \sim O(g^{\frac{4}{3}}\eta^{\frac{1}{3}}V^{-2}).$$

The small power dependence on η implies a surprisingly rapid growth. For this mode there is a slip layer of small, but finite thickness $\alpha^{-1} \sim k^{-1}R^{-\frac{2}{3}} \sim \eta^{\frac{2}{3}}g^{-\frac{1}{3}}$ in which the field and conducting fluid cease to be effectively tied and interchange of the two fluids is possible.

The increasing growth rate as $G \to 0$ for very short waves goes as

$$|\omega| \sim (kg)^{\frac{1}{2}} \to \infty.$$

This is simply explained by considering the 'magnetic Reynolds number', which for our purpose can be defined as $|\omega|/\eta k^2$, which is then of order

$$|\Omega|G^2R \sim O(R^{-2})$$

and is very small. The fluid, being virtually a non-conductor, is simply Rayleigh–Taylor unstable. Of course, it is at such short wavelengths that the neglected viscosity must play a part in limiting the rapid growth rate. A criterion of the significance of viscosity is obtained when the viscous Reynolds number Re of the magnetic slip layer of thickness α^{-1} becomes of order unity, i.e.

$$\mathrm{Re} = |\omega|/\alpha^2\nu \sim g^{\frac{2}{3}}\eta^{\frac{5}{3}}/\nu V^2 \sim 1,$$

or

$$\eta/\nu \sim (V^3/g\eta)^{\frac{2}{3}} \sim R^{\frac{2}{3}},$$

where ν is the kinematic viscosity, assumed isotropic.

No overstable modes have been found in this problem in contrast to Jukes (1961), but there are in both (*a*) and (*b*) oscillatory modes damped at a rate $\propto R^{-\frac{1}{2}}$.

5. Conclusion

Inclusion of small but finite resistivity η into the MHD equations is sufficient in this particular example to introduce entirely new branches of solutions in comparison with idealized MHD in which $\eta \equiv 0$. In this modified Rayleigh–Taylor problem what appeared to be stable, moderately long waves according to idealized theory are now found to grow aperiodically and quite rapidly at a rate $\propto \eta^{\frac{1}{3}}$. There are no overstabilities, but only damped oscillations in addition.

Finally, it must be emphasized that this is a specific result for a particular and simple model. Generalization to more complicated configurations of conducting fluids including pressure gradients, such as the pinch, is a subject for future work.

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